ALGEBRA AND TOPOLOGY
HOMEWORK THREE
DUE: 8/1

There are two different types of labels: alphabets and numbers. You only need to write up your solutions to those exercises labelled by alphabets. The rest is for fun.

Exercise A. Let K be the Klein bottle defined in class. Show that

$$K \cong \mathbf{R}P^2 \# \mathbf{R}P^2.$$

Exercise B. Let (S, τ) be a topological space and $T \subseteq S$ be a subset. Consider the collection of subsets

$$\sigma := \{V \cap T \mid V \in \tau\}.$$

Show that σ defines a topology on T for which the inclusion map $\iota\colon T \to S$ is a continuous transformation. Show that the topology is *minimum* in the following sense: if (T, σ') is a topology making ι continuous, then for each $V \in \tau$, there exists a neighborhood $N \in \sigma'$ such that $N \subseteq V \cap T$.

Exercise 1. A hex board is an array of regular hexagons arranged into a diamond shape in such a way that there is the same number of hexagons along each side of the board. The game of hex is played on a hex board in the following manner. Two players alternate and the aim of the game is to connect pairs of opposite sides. More precisely, if the players are labelled A and B then one pair of opposite sides is labelled A and the other B. In turn each player labels a previously unlabelled hexagon with her or his symbol. The winner is the player who first obtains a connected path of adjacent hexagons stretching between the sides of that player's label. Here is a picture for a hex board (cf. Figure 1).

Show that there is always a winner.

Exercise 2. Let S be a finite set and consider $E := 2^S \setminus \{\emptyset, S\}$ the poset of all non-empty proper subsets, ordered by containment. Show that if an order-preserving map $f\colon E \to E$ does not have a fixed point then it is surjective, and hence an automorphism.

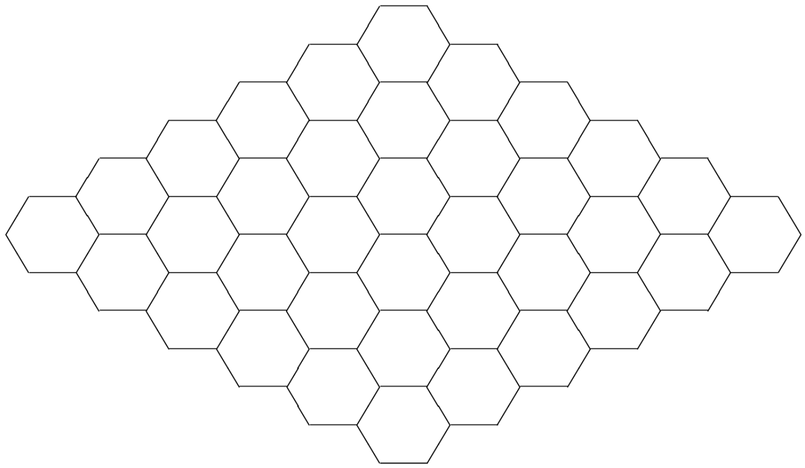

Figure 1. A hex board.