ALGEBRA AND TOPOLOGY
HOMEWORK EIGHT
DUE: NONE

You don't have to hand in Homework Eight. Instead, I encourage you to work on and focus on your research project. I am giving you the homework only for the sake of completeness of the lecture.

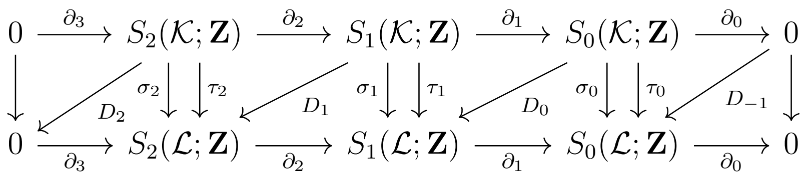
Exercise 1 (Invariance)**.** This exercise is about the invariance of the group homomorphisms induced from simplicial approximations. The aim is to prove the following theorem.

Theorem 0.1 (Invariance Theorem)**.** *Let $f\colon S \to T$ be a continuous transformation and $\tau\colon \mathcal{K} \to \mathcal{L}$ be a simplicial approximation for f. Then the induced map $\tau_k\colon \mathrm{H}_k(S;\mathbf{Z}) \to \mathrm{H}_k(T;\mathbf{Z})$ is the same for all simplicial approximations of f and for every k.*

(a) Let τ and σ be two simplicial approximations of f from the same triangulation $\mathcal{K}$ of S to the same triangulation $\mathcal{L}$ of T. Then τ_k and σ_k are homologous, i.e. for every k-chain $\Delta \in S_k(\mathcal{K};\mathbf{Z})$ there exists a $(k+1)$-chain $\tilde{\Delta} \in S_{k+1}(\mathcal{L};\mathbf{Z})$ such that

$$\tau_k(\Delta) - \sigma_k(\Delta) = \partial_{k+1}\tilde{\Delta} \in S_k(\mathcal{L};\mathbf{Z}).$$

Hint: Consider the diagram



Construct the maps D_k as indicated above with the property that

$$\tau_k - \sigma_k = \partial_{k+1} \circ D_k + D_{k-1} \circ \partial_k. \tag{0.1}$$

Certainly, $D_{-1} = D_2 = 0$. You can construct D_0 via

$$D_0(P) := \overline{\tau(P)\sigma(P)}.$$

How do you construct D_1? If (0.1) holds, show that τ_k and σ_k give the same map from $\mathrm{H}_k(\mathcal{K};\mathbf{Z})$ to $\mathrm{H}_k(\mathcal{L};\mathbf{Z})$.

(b) Let $\mathcal{K}^+$ be the barycentric subdivision of $\mathcal{K}$. Consider the identity transformation $\iota\colon \mathcal{K}^+ \to \mathcal{K}$ by regarding both sides as topological spaces. Find a simplical approximation of ι and show that it induces an isomorphism from $\mathrm{H}_k(\mathcal{K}^+;\mathbf{Z})$ to $\mathrm{H}_k(\mathcal{K};\mathbf{Z})$ which inverts the group homomorphism from $\mathrm{H}_k(\mathcal{K};\mathbf{Z})$ to $\mathrm{H}_k(\mathcal{K}^+;\mathbf{Z})$ induced by subdivisions.

(c) Let S, T, and U be topological spaces with triangulations $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$. Suppose $f: S \to T$ and $g: T \to U$ are two continuous transformations with simplicial approximations τ and σ. Show that $\mu := \sigma \circ \tau: \mathcal{K} \to \mathcal{M}$ is a simplicial approximation of $g \circ f$ and

$$\mu_k = \sigma_k \circ \tau_k$$

for every k.

(d) Conclude the theorem using (a) and (b) and the fact that any two triangulartions of a sequentially compact surface have a common refinement.

The conclusion of Theorem leads to the following definition.

Definition 0.2. Let $f: S \to T$ be a continuous transformation between surfaces with S sequentially compact. Define

$$\mathrm{H}_k(f): \mathrm{H}_k(S; \mathbf{Z}) \to \mathrm{H}_k(T; \mathbf{Z})$$

to be the group homomorphism $\mathrm{H}_k(\tau)$ for any simplicial approximation τ of f.